5.2 – Diagonalization

Definition: A square matrix A is said to be **diagonalizable** if it is similar to some diagonal matrix; that is, if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal. In this case, the matrix P is said to **diagonalize** A.

Theorem 5.2.1 If A is an $n \times n$ matrix, then A is diagonalizable if and only if A has n linearly independent eigenvectors.

#5 Find a matrix P that diagonalizes A, and check your work by computing $P^{-1}AP$.

$$A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$$

#17 Compute the matrix A^{10} by first diagonalizing A.

$$A = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$$

Theorem 5.2.3 If k is a positive integer, λ is an eigenvalue of a matrix A, and $\mathbf{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector.

#9 Let $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$.

a. Find the eigenvalues of A.

b. For each eigenvalue λ, find the rank of the matrix $\lambda I - A$.

c. Is A diagonalizable? Justify your conclusion.

If λ_0 is an eigenvalue of an $n \times n$ matrix A, then the dimension of the eigenspace corresponding to λ_0 is called the **geometric multiplicity** of λ_0, and the number of times that $\lambda - \lambda_0$ appears as a factor in the characteristic polynomial of A is called the **algebraic multiplicity** of λ_0.

Theorem 5.2.4 Geometric and Algebraic Multiplicity

If A is a square matrix, then:

a) For every eigenvalue of A, the geometric multiplicity is less than or equal to the algebraic multiplicity.

b) A is diagonalizable if and only if its characteristic polynomial can be expressed as a product of linear factors, and the geometric multiplicity of every eigenvalue is equal to the algebraic multiplicity.

#14 Find the geometric and algebraic multiplicity of each eigenvalue of the matrix A, and determine whether A is diagonalizable. If A is diagonalizable, then find a matrix P that diagonalizes A, and find $P^{-1}AP$.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$$

Theorem 5.2.2

a) If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of a matrix A, and if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are corresponding eigenvectors, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a linearly independent set.

b) An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Similarity

Definition: A transformation of the form $A \rightarrow P^{-1}AP$ is called a **similarity transformation**.

Definition: If A and B are square matrices, then we say that **B is similar to A** (or that A and B are **similar matrices**) if there is an invertible matrix P such that $B = P^{-1}AP$.

Example: Prove that if A and B are similar matrices, then $\det(A) = \det(B)$.

Any property that is preserved by a similarity transformation is called a **similarity invariant** and is said to be **invariant under similarity**.